Targeting Informative Messages to a Network of Consumers

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Abstract

This paper considers duopolists targeting informative messages to consumers who share information locally with their network neighbors. A monopolist targets a parsimonious set of nodes that informs all consumers either directly or by word-of-mouth. A duopolist faces a tradeoff between this efficient targeting and possible preemption by a competitor's message. Under gentle price competition, duopolists saturate the network when messages are cheap, and target sets similar to the monopolist's when messages are costly. Under fierce price competition, duopolists' messages segment the network in an intermingled patchwork. Effects of network structure and the cost of messages on firm outcomes are discussed.

JEL: D83, D85, L13, M31

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1 Introduction

Word-of-mouth is an important way in which consumers can learn about a product, and there is evidence that the existence and nature of word-of-mouth have a tangible impact on consumer behavior (Godes and Mayzlin (2004), Godes and Mayzlin (2009)). Communication partners, however, are not selected at random. The structure of such communication - who talks to whom - is therefore of interest to a firm or other entity that would like to spread information among those consumers: when some consumers learn a piece of information, it can be amplified in a particular way by retransmission to others they interact with, and so informing a given set of people need not require *directly* informing them. In particular, the problem for a direct marketer who can send messages to consumers is affected by the extent and structure of word-of-mouth among the population who will receive the messages. In this spirit, Zubcsek and Sarvary (2011) considers the problem for a firm (or firms) that can choose a number of direct marketing messages to send to a population of consumers who are arranged in some network whose links capture channels along which information can flow; Galeotti and Goyal (2009) considers both this case and the case in which the firm can target individuals of specific degree. These contributions uphold the notion that incorporating information on the structure of the network enables the firm to improve the efficiency of its marketing strategy.

However, as technology advances and the structure of communication networks becomes ever more observable, the targeting of such campaigns can become still more specific - and its consequences more tangible. The graph of connections among users of Twitter, for example, is, despite its complexity, public knowledge; the retransmission through 'retweeting' of a message sent by a business to its followers can be readily observed. Rather than simply deciding how many messages to send to potential customers, knowing that an opaque process of word-of-mouth could follow, marketers are gaining an unprecedented ability to send messages to precisely chosen locations in a well-defined communication network. Indeed, early in the life of Twitter, businesses enthusiastically adopted it as a marketing tool;[1] business schools including those at Harvard and Columbia now offer courses in social media marketing.[2] There

[1] "Marketing Small Businesses With Twitter" by Claire Cain Miller, *New York Times*, July 22, 2009

[2] "B-Schools All A-Twitter Over Social Media" by Sommer Saadi, *Bloomberg Businessweek*, July 26, 2010

are surely a variety of motivations for the use of social media by a marketer: to engage customers as de facto spokespeople for a product or to engender loyalty, for example. One particular motivation that this paper will focus on is linked to the increasing transparency of communication: if a firm seeks to disseminate some new piece of information and can *observe* the graph of communication channels among consumers, to which individuals should it send that information?

This question of targeted dissemination is familiar from an extensive literature on the targeting of 'influentials' or 'opinion leaders' in marketing a product, dating at least to the seminal study of the targeting by pharmaceutical marketers of influential physicians by Coleman, Katz, and Menzel (1966).[3] Following Rogers and Cartano (1962), Iyengar, Van den Bulte, and Valente (2011) taxonomizes the identification of such individuals as possible via self-designation, other-designation or 'sociometric' techniques designed to reverse-engineer the structure of network of influence to identify 'central' individuals. The model below speaks to this third means; consumers are assumed homogeneous except for their position on the network in order to isolate the effect of network structure on the pattern of who should be targeted directly in various competitive environments.

These targeting approaches are naturally constrained by the fact that a network graph is highly complex, and so identifying the 'right' locations to target is correspondingly difficult. Indeed, a rich literature in computer science has considered the closely related topic of the value of capturing a consumer when they can go on to influence others recursively across network links (Domingos and Richardson (2001), Richardson and Domingos (2002), Kempe, Kleinberg, and Éva Tardos (2003)); such problems in graph coverage are known to be NP-hard (Garey and Johnson (1979)). This paper proposes a theoretical model that captures key features of the dissemination problem in situations in which communication patterns in the relevant network are understandable by the decision-maker. Settings that can be literally captured by the model developed in the present paper must therefore operate over 'small' networks in order to be tractable. However, this can mean a small number of individuals (as in the Coleman, Katz, and Menzel (1966) study which constructed a picture of the influence

[3] A question re-analyzed in Van den Bulte and Lilien (2001), which considers in detail the confounding effects of advertising volume on the conclusion of the original study.

network among physicians local to the target area), or, exploiting that the model can naturally be interpreted in the abstract, interpreting the network with a coarser definition of a node. For example, say a software company is marketing a new piece of software that is designed for video game designers. The *firm-level* network of collaborative relationships among game developers could capture for the software company relevant channels of communication along which news of its product will spread from users to the uninformed. Then the number of individuals in the relevant market may be large, but a practically relevant and tractable communication network can nevertheless be identified.

In the model, a firm (variously a monopolist or one half of a duopoly) seeks to disseminate some piece of information, perhaps about a special offer or a new product, to a population of initially uninformed consumers,[4] who are arranged in a publicly observable social network. The firm can choose a set of consumers to whom it will send (costly) direct marketing messages containing the piece of information. There are two ways for individual consumers to learn: a consumer learns the information today if the firm directly informs the consumer, or learns the information tomorrow if they are linked in the network to someone who was directly informed.

Several contributions that analyze similar settings to this model raise doubts that it is best to target the most connected individuals (for example Tucker (2008), Watts and Dodds (2007)), and these doubts are further validated here. For the case of a monopolist, a strategy that targets the most connected individuals overlooks any weakly-interconnected regions of the network; yet, a strategy that sends messages to randomly chosen locations does not acknowledge that well-connected individuals can generate the most word-of-mouth. To maximize the impact of direct marketing messages instead requires a more subtle targeting strategy that acknowledges both the outsized influence of the well-connected *and* the existence of more barren regions. In the focal case of the model in which information travels at most one degree from its recipient, the monopolist does best by targeting messages to the set of nodes corresponding to the graph-theoretic concept of the *minimum dominating set*[5]. Targeting this set is the cheapest way to ensure that all consumers are eventually in-

[4]Advertising here therefore performs a role as in (Butters 1977), and 'social influence' operates purely as information transfer, as set forth in Katz and Lazarsfeld (1955).

[5]This is a set of nodes such that all nodes in the graph are either in the set or are direct neighbors to a

formed and so maximizes the impact of a given number of messages, but it is generally very different than targeting the well-connected. The spirit of this result is readily generalizable upward to richer assumptions on the prevalence and distance of information transmission. One implication is that even when individuals differ only in their network position, those who appear ex-post 'influential' - as a result of being targeted first - are located in diverse, almost idiosyncratic positions that can be sensitive to small changes in network structure.

The competitive case brings a new tension for firms. There remains an incentive to rely on word-of-mouth rather than on costly direct marketing, but there may also be an incentive to avoid an individual learning about the competitor's product first. To assess the impact of this competitive pressure, below the competitive case is further subdivided into two according to the extent to the loss for one firm associated with an individual knowing also of the other firm. In both cases the firm does better whenever it reaches a consumer first, but the value obtained from reaching a consumer at the same time as a competitor can be variously half of the first-arriver value or nil.

When competitive pressure is high, the cost of sharing the market outweighs the benefit to choosing the word-of-mouth-maximizing set, and firms are driven to send messages to distinct regions of the network. However, the existence of word-of-mouth means that this segmentation takes a specific form, which in the two-period case corresponds to the firms sending messages to *disjoint dominating sets* of the network graph. This means that the two firms segment the network in an overlapping 'patchwork' rather than, say, dividing the network east-west. This is because an east-west division would give either firm incentive to send fewer direct messages and exploit word-of-mouth, and thus in turn give its competitor incentive to preempt them in that region. In this sense competitive pressure damages each firm's ability to exploit word-of-mouth, since doing so would expose them to preemption.

In the case with lower competitive pressure, the cost of sharing the market is low relative to the benefit of targeting the set of consumers most efficient in generating word-of-mouth, so equilibria can be symmetric, but their nature depends on the cost of sending a message. When the cost of marketing is high, the firms restrict their marketing volume in a similar way as would a monopolist in order to exploit word-of-mouth. In particular, neither informs

node in the set.

a *redundant set* of consumers, and each consumer hears of at least one firm either directly or indirectly. This means that no message is sent purely in order to capture the consumer that receives it; each message must have a word-of-mouth effect that is irredundant to that generated by other messages. If competitive pressure is low and the cost of sending a message is low, sending a message that captures only the consumer that receives it is profitable even if it means 'sharing' that consumer. There is then an 'arms-race' effect: in the unique equilibrium sees both firms engage in a mass-marketing campaign and send messages to everyone. Since equilibrium selection depends on the cost of sending a message, firm profits in this case can fall as the cost of sending messages falls: only a high cost of targeting can temper the incentive to send extra messages to preempt a competitor rather than rely on word-of-mouth.

In all settings extra links in the network are never bad for the firm in equilibrium. However, the extra links have a positive marginal benefit only in the cases in which the firm is able to exploit word-of-mouth in equilibrium; that is, only when the firm is a monopolist or when competition is low and the cost of messages high enough to preclude an advertising arms race. The intuitive notion that more communication among consumers can facilitate more efficient targeting is thus confirmed only when the nature of competition in the targeting game does not undermine the firm's ability to exploit word-of-mouth.

The case in which potential price competition drives perfect segmentation has parallels in the partly related settings considered in Banerji and Dutta (2009), Roy (2000) and Galeotti and Moraga-Gonzalez (2008). Banerji and Dutta (2009) consider a model of network externalities in a market with two producers. Their framework assumes network benefits to adopting a technology are present only across links in the network graph and not over the whole population, and find that local network externalities permits strong market segmentation and therefore positive profits, even when the firms are a priori identical and are Bertrand competitors. Roy (2000) analyzes a model in which two firms first target information to a set of consumers and then engage in price competition, finding that perfect segmentation and pure local monopoly emerges. The present model confirms a similar outcome for the case with word-of-mouth, with the associated particular segmentation structure. Galeotti and

Moraga-Gonzalez (2008) considers a Bertrand-like environment in which the market has two segments, and demonstrates that if there is sufficient variation in the cost to access each segment, duopolists can earn positive profits in equilibrium. This result has particular parallels with the outcome in the most Bertrand-like of the cases considered in this paper: a similar conclusion holds when the pattern of segmentation is itself an outcome of the firms' strategies. Another related contribution which considers a competitive targeting problem is Iyer, Soberman, and Villas-Boas (2005), where consumers are heterogeneous in their idiosyncratic preference for each firm's product, and targeting is by type.

The remainder of the paper proceeds as follows. Section 2 outlines the framework of the model. Section 3 discusses the monopolistic case; its Proposition 1 establishes the solution to the monopolist's problem and Corollary 1 performs comparative statics on the number of connections in the network and on costs. Section 4 discusses the duopolistic case. Proposition 2, establishes equilibria in the case with high intensity of competition, and Propositions 3 and 4 equilibria in the case with low intensity of competition, with comparative statics respectively in Corollaries 2 and 3. Section 5 discusses how the results for each competitive regime look in some simple families of networks. Section 6 considers implications of the suggestive results of the model for the structure of industries that function as information intermediaries between firms and consumers. Section 7 concludes.

2 Model

There is a set $A = \{1, ..., n\}$ of consumers.[6] Each consumer is a node in an undirected and connected[7] graph (A, g), where g is a real-valued $n \times n$ matrix in which g_{ij} represents the relationship between consumers i and j.[8] The graph represents the *social network* across which information can flow among consumers. $g_{ij} = 1$ if there is a link between i and j and 0 otherwise; since the graph is undirected $g_{ij} = g_{ji}$ for all i and j, and $g_{ii} = 0$ for all i. The

[6]As mentioned above, the setting in question will define the level at which a network can be defined over the relevant market. Depending on the setting, we may thus interpret these "consumers" as individuals or as larger clusters.

[7]That the graph is connected is without loss of generality, since the analysis of an unconnected graph will be identical to the separate analysis of its connected components.

[8]Notation for the graph follows convention, as in Jackson (2008).

network size is n, and since A is assumed fixed, for convenience denote the graph g.

The *open neighborhood* of consumer i is $N(i) = \{j : g_{ij} = 1\}$, the set of consumers that i is linked to. The *closed neighborhood* of i is $N[i] = N(i) \cup \{i\}$. Let $S \subseteq A$ be some set of consumers and let $s = |S|$ be its cardinality. The open neighborhood of S is $N(S) = \cup_{i \in S} N(i)$. Let $\Omega(S) = N(S) - S$; this is the set of consumers in the neighborhood of S who are not themselves in the set S. For notational consistency, let $\Omega(i) \equiv N(i)$. Following this, let $\omega(i) = |\Omega(i)|$ denote the number of neighbors to consumer i (equivalent to i's degree), and similarly let $\omega(S) = |\Omega(S)|$. Some $l \in S$ has a *private neighbor* outside S if there is some vertex m in the open neighborhood of l that is not in the open neighborhood of any other vertex in S (following, for example, Fellows, Fricke, Hedetniemi, and Jacobs (1994)).

There are two firms indexed $k = 1, 2$ which produce a homogeneous product at zero cost (below we also consider a baseline case in which there is only a single firm). Let there be two periods, $t = 1, 2$. Initially consumers do not know that either firm's product exists, but firms have the chance to inform consumers: each firm can observe the social network g and simultaneously chooses at $t = 1$ a set of consumers $S_k \subseteq A$ to whom it will send a *direct marketing message* at a cost c per message.[9] All $i \in S_k$ (those consumers who receive a direct marketing message from firm k) learn at $t = 1$ that firm k's product exists. All consumers $i \in \Omega(S_k)$ (those consumers who are linked to at least one consumer who received a direct marketing message) learn at $t = 2$ that firm k's product exists. This implies non-optional and costless information transmission along links of the social network. Information travels only one degree, mirroring Zubcsek and Sarvary (2011).

The payoff earned by firm k depends on (i) the number of consumers who learn about its product, and (ii) whether those consumers also know about the other firm's product. Assume that the firm k earns some value v from a consumer who learns of k before she learns of l, and zero from a consumer who never learns of k. The analysis below considers separately two cases: one in which firm k earns value $\frac{1}{2}v$ from a consumer who learns of both firms simultaneously, and one in which firm k earns a value 0 from such a consumer. One motivation for the former case is that two firms 'share' consumers who know about both

[9] The constant cost per message follows Zubcsek and Sarvary (2011) and is a conservative assumption; in many settings this will be convex in s_k.

firms; one motivation for the latter case is that two firms compete destructively in some way over consumers who know about both firms. These two cases thus follow the parameter in Zubcsek and Sarvary (2011) representing 'intensity of competition'. The general implication in either case is that the extent to which the populations that know of each firm's product overlap affects each firm's payoff.

3 The monopolistic case

First consider the case in which there exists only a single firm m. Let the firm earn a payoff according to the number of consumers who learn at any time about the product, so that when the monopolist chooses a set S_m, the firm's payoff is given by

$$\pi(S_m) = (s_m + \omega(S_m))v - s_m c. \tag{3.1}$$

Assume $v > c$. The solution to the monopolist's problem is given in the following result:

Proposition 1. *A payoff-maximizing S_m^* for the monopolist is a* minimum dominating set *of g. That is:*

$$S_m^* \cup \Omega(S_m^*) = A \tag{3.2}$$

$$\nexists S' : S' \cup \Omega(S') = A, s' < s_m^* \tag{3.3}$$

The proof (and others to follow) appears in Appendix A. 3.2 says that S_m^* is a *dominating set* of the graph g. This is some set so that all consumers are either in the set or its neighborhood. 3.3 says that S_m^* is a *minimum dominating set* of g. This means that there does not exist a dominating set with a lower cardinality than S_m^*. The result therefore states that the best strategy for the monopolist is to choose the smallest (equivalently cheapest) set of consumers such that all consumers will learn about the firm's product. The cardinality of such a set in a given network g is called the *domination number* of the graph and is denoted $\gamma(g)$; the number of messages sent by a monopolist is therefore bounded above by this number. By Ore (1962), the domination number of a connected graph is at most half of the number of vertices, but in general the bound will be tighter. Denote (for later reference) the set of minimum dominating sets in g by $MDS(g)$.

Two implications of this result are that the ability to perfectly target is valuable, in the sense that (i) the firm does better than in the case without any targeting, in which it chooses a number of messages to send to random nodes, and (ii) the firm does better than in the case with targeting based on some non-locational statistics like the degree of each node. To see the spirit of (i), consider a network structure as in Figure 1. In this network there are

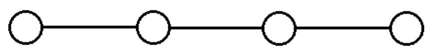


Figure 1: Line network

four minimum dominating sets: four ways to locate two messages such that all consumers learn about the product. When the monopolist targets such a set, it pays $2c$ and earns $4v$. If, however, it cannot perfectly target messages and sends two messages at random, the firm again pays $2c$ but now earns (in expectation) only $\frac{11}{3}v$, since there is a probability $\frac{1}{3}$ that the messages arrive so that one consumer never learns about the product. The wedge between the payoff to optimal targeting and random targeting represents a premium that the firm would be willing to pay to discover the precise network structure, or for a service that could perfectly target messages over a service that randomly targeted messages.

On (ii), consider the network in Figure 2. Sending messages to the minimum dominating

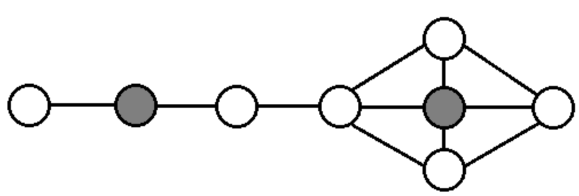


Figure 2: Network with minimum dominating set marked

set (the shaded nodes) results in all eight consumers learning about the firm's product at a cost of only 2 messages. Say instead that the firm sent the same number of messages but used a rule-of-thumb that specified targeting the most well-connected consumers. Then the firm would send messages to two of the consumers with four neighbors; this results in at most six consumers learning about the firm's product. This structure for optimal influence is general to any network structure that has, loosely speaking, both dense and barren areas: targeting only the best-connected will over-inform the dense regions and under-inform the barren regions relative to the strategy in Proposition 1 (Section 5 below discusses this in relation to 'small world' networks). Targeting that maximizes the impact of a given number of messages cannot generally be achieved by such rules as targeting only the best-connected. Therefore, when targeting is possible, knowing the precise network structure is more valuable than knowing some summary statistic like the degree of each node.

Finally, an implication of Proposition 1 is that the monopolist does better when the domination number of g is smaller. The domination number is sensitive to the precise structure of the graph, but we can nevertheless demonstrate the following:

Corollary 1. *In the monopolistic case,*

a. *the monopolist's profit is weakly increasing (equivalently, the cost of optimal targeting is weakly decreasing) in the number of links in g,*

b. *the marginal benefit to the monopolist of adding a link to g is either zero or c, depending on the location of the link, and*

c. *the monopolist's profit is decreasing in the cost c of sending messages.*

Part c. is immediate, since as long as $v > c$ the monopolist's optimal targeting does not depend on c. Parts a. and b. follow immediately from the known results that the domination number is never increasing in the number of edges in a graph, and decreases by at most one when an edge is added (see, for example, Haynes and Henning (2003)). This validates the intuitive notion that targeting can be more efficient when there is more communication among consumers, although again the precise structure of g and the location of the marginal link are relevant to knowing its value. While this is true for a marginal link, it is, however, not the case that networks with more links have lower domination numbers than those with fewer links; in Section 5 we illustrate this in a 4-consumer example.

4 The duopolistic case

Now return to the duopolistic case. The two firms $k = 1, 2$ observe g and simultaneously choose a set S_k of consumers to whom they will send direct marketing messages. As outlined above, we will consider separately two models that differ in the intensity of competition between the two firms over 'common' consumers - those who learn about both firms simultaneously.

4.1 High intensity of competition over common consumers

First we consider a setting in which the intensity of competition is 'high': firm k earns a value v per consumer who learns about firm k before they learn about firm l, and 0 per consumer who either learns of both firms simultaneously or does not learn about firm k at all. This captures a situation in which a firm benefits from getting to a given consumer first, but that competition nullifies this benefit when the firms arrive at the same time.

The following result characterizes Nash equilibria in this one-shot, simultaneous move game played between the two firms.

Proposition 2. *In the high intensity of competition case, the pair (S_1, S_2) is a Nash equilibrium in pure strategies if and only if S_1 and S_2 are disjoint dominating sets of the network graph and $S_1 \cup S_2 = A$.*

This outcome is one of perfect segmentation: although by $t = 2$ all consumers have learned of both firms, each firm enjoys one period of sole capture of those consumers to whom it sent messages. This is not surprising: the firms will never each send a message to any common consumer in equilibrium, since then both pay c and receive 0 for that consumer; also, all consumers must receive one message in equilibrium, since if not any one firm could earn v and pay only c to send a message to such a consumer.

However, the segmentation has a striking feature: the firm's chosen targets are intermingled throughout the network. Consider again the simple line network. Figure 3a shows a 'geographic' segmentation pattern that has each firm taking a 'side' of the network: one firm sends messages to the shaded consumers, the other to the unshaded; this is not an equilib-

(a) No equilibrium (b) Equilibrium

Figure 3: Illustrating Proposition 2

rium. Figure 3b shows a different segmentation pattern (disjoint dominating sets) that does constitute an equilibrium.[10] An important feature of this equilibrium pattern is that each consumer learns of one firm today and of the other firm tomorrow by word-of-mouth.

Why should this be? This is an interplay of the two competing incentives for each firm. Neither firm will send a message to a consumer that would otherwise have learned of the firm by word-of-mouth before the other firm reached them: this is the opportunity afforded by word-of-mouth to reduce direct marketing effort. All consumers must receive some message, however, since intense competition means that is is certainly better for a single firm to send a message that results in sole capture of a consumer than to leave that consumer unserved *or* to 'share' that consumer by word-of-mouth. Informally, word-of-mouth drives each firm to be parsimonious with its messages in a given local area of the network, but this parsimony drives the competing firm to send messages that fill in these gaps. The paradoxical consequence is that both firms are unable to exploit word-of-mouth in equilibrium, since in doing so they are exposed to preemption.

To see this concretely, observe that in the case of Figure 3a, either firm could reduce the number of messages it sends by 1 but still remain the first firm to reach 2 consumers; since this yields a higher payoff than the proposed pattern of segmentation, such a pattern is not an equilibrium. Word-of-mouth thus polices segmentation: in the equilibrium pattern in Figure 3b, neither firm can reduce the number of messages it sends without giving up the first shot at a consumer. This result therefore demonstrates that the existence of the incentive given by word-of-mouth to reduce the number of direct messages sent can lead to patterns of segmentation across communication networks that appear somewhat different from a 'simple'

[10]Not that this is certainly not unique, since there are many pairs of disjoint dominating sets in this example.

local monopolization pattern.

Disjoint dominating sets can in general be unequal in cardinality, and so an implication of Proposition 2 is that in equilibrium the outcomes for firms can be highly asymmetric. This parallels a similar result in the duopoly with fixed market segmentation in Galeotti and Moraga-Gonzalez (2008). Along these lines Section 6 below discusses potential implications of this suggestive result for the type of information intermediaries that competing firms may seek to employ to disseminate their messages when the structure of their market corresponds to this case. For the firms, as we would expect, industry profits are lower here than in the monopolistic case. Segmentation means that revenue per consumer is higher, but competitive pressure drive the firms to a greater combined advertising volume than the parsimonious monopolistic solution.

As in the monopolistic case we can explore how firm outcomes depend on the number of links in the network and on the cost of sending messages:

Corollary 2. *In the duopolistic case with high intensity of competition,*

a. *average profit per firm in equilibrium is unchanging in the number of links in g,*

b. *adding a link to g weakly increases the potential asymmetry of firm outcomes in equilibrium, and*

c. *each firm's profit is decreasing in the cost c of sending messages.*

Part a. reflects that on average each firm ultimately serves half of the consumers in A at the monopolized value v, no matter what the precise structure of the network, since segmentation is perfect in equilibrium. However, part b. derives from the fact that the asymmetry in firm outcomes can be more pronounced when there exist pairs of disjoint dominating sets that are more unequal in cardinality - that is, when the domination number of g is lower. Again a higher cost of sending messages does not change the character of equilibria and so higher costs feeds directly to lower profit for each firm in a given equilibrium.

4.2 Low intensity of competition over common consumers

Second, consider a case in which the nature of competition is different. Again let v be the value to firm k per consumer who learns first about firm k, and again let the value to firm k be 0 per consumer who learns about firm l first. Now, however, assume that firm k earns $\frac{1}{2}v$ per consumer who learns about both firms at the same time. For example, say that the two firms are promoting essentially identical email services. To a consumer there is no relevant difference between the two, so she adopts whichever he hears about first and chooses at random when she hears about both together. Since the product is free to the consumer, a firm receives the same value whether from that subscriber whether she chose that firm while ignorant of the other or at random.

The following result demonstrates that in this case, when the cost of sending marketing messages is sufficiently low, the unique equilibrium features both firms engaging in mass-marketing and sending a message to every consumer:

Proposition 3. *In the low intensity of competition case, if* $c < \frac{1}{2}v$ *the unique Nash equilibrium in pure strategies for any network* g *is* $S_1 = A$, $S_2 = A$.

Intuitively, for low cost of direct marketing, the net benefit of relying on word-of-mouth rather than directly informing a consumer and the net loss of sharing consumers with the competitor are both small relative to the loss associated with being preempted by the competitor, and so a marketing arms-race drives both firms to saturate the network with messages. One implication of this result as contrasted with Proposition 2 is that when the cost of sending messages is low, each firm may earn a greater profit when price competition is high.

For the case in which the cost of sending a message is above the threshold in Proposition 3 we will require some further definitions for the network g. Denote by $ID(g)$ the set of *irredundant sets* in g, defined as those sets of vertices such that no vertex can be removed from the set without reducing the number of vertices in the closed neighborhood of the set (Bollobás and Cockayne (1979)). Denote by $IR(g)$ the *upper irredundance number*, defined as the maximum cardinality of an irredundant set in g (Cockayne, Hedetniemi, and Slater (1978)). A set that is not irredundant is a *redundant set*.

Further, denote by $OID(g)$ the set of *open irredundant sets* in g, defined as those irredundant sets S such that each $i \in S$ has at least one private neighbor outside the set S (Farley and Schacham (1983), Fellows, Fricke, Hedetniemi, and Jacobs (1994)). Similarly, denote by $OIMDS(g)$ the set of *open irredundant minimum dominating sets* in g, defined as those open irredundant sets that are also minimum dominating sets (such a set always exists in a connected graph (Bollobás and Cockayne (1984)).

(a) Redundant set (b) Irredundant set (c) Open irredundant set (d) Open irredundant MDS

Figure 4: Illustrating classes of irredundant set

Figure 4 illustrates examples of these notions of irredundance for the 4-consumer line network. Note that in 4d each consumer is in the closed neighborhood of the set (so the set is dominating), but if either of the shaded nodes is dropped from the set, then one of the unshaded consumers is no longer in the closed neighborhood of the set (that is, each shaded consumer had an unshaded private neighbor: the set is open irredundant).

Then the following result characterizes equilibria when the cost of sending a message is sufficiently high:

Proposition 4. *In the low intensity of competition case when* $c > \frac{1}{2}v$*:*

a. *In equilibrium neither firm chooses a redundant set, and so each firm sends at most* $IR(g)$ *messages.*

b. *In equilibrium each consumer learns of at least one firm by tomorrow.*

c. *There is a symmetric equilibrium at* $S_j = S_k = S$ *if and only if* S *is an open irredundant minimum dominating set.*

Leaving aside the technical details of the graph-theoretic definitions, the intuition for these results is the following: when the cost of direct marketing is high, the net benefit of relying on word-of-mouth rather than directly informing a consumer is high, and so both firms are driven to reduce their direct marketing volume to exploit word-of-mouth. In equilibrium, each message sent by a firm must yield either first shot at at least one consumer, or a simultaneous shot at at least two consumers. The most important contrast with the fiercely competitive setting is that word-of-mouth is relevant in equilibrium, in the sense that it can be that some consumers do not receive direct messages from either firm but learn of at least one firm tomorrow by communication with fellow consumers. This is because symmetric equilibria (and more generally equilibria with overlap) are sustainable for two broad reasons. First, the cost of sharing a consumer is lower than before. Second, the competitor does not necessarily have incentive to plug a gap left by one firm in its effort to exploit word-of-mouth; a consumer shared by word-of-mouth is more profitable than sending an extra message today to monopolize that consumer directly.

Part a. of Proposition 4 states that neither firm chooses a set of consumers such that informing one fewer consumer affects only the payoff obtained over that consumer; in this sense each direct marketing message must be associated with some (at least potential) word-of-mouth effect in equilibrium. Part b. says that no consumer is left out of both the messages and word-of-mouth, which follows directly from the primitive assumption that v, the value to a firm of sole capture of a consumer, exceeds c, the cost of sending one message. Part c. is in some sense a combination of a. and b., and says that at least one symmetric equilibrium exists in any network, and in a symmetric equilibrium, all consumers hear of both firms either by direct message or word-of-mouth. This dominating set is parsimonious in that it is a minimum dominating set, and respects the irredundancy condition in a. in that is open irredundant.

The role of irredundancy in Proposition 4 is similar in spirit to the selection of a minimum dominating set by a monopolist, but in a sense stronger since not all minimum dominating sets are open irredundant (see Figure 5 below). In this competitive case, it is again true that neither firm can send one fewer message without some consumer not learning either directly or indirectly the firm's information. The rationale for this strategy is identical: in both this and the monopolistic case the incentive to exploit word-of-mouth is sufficiently strong to drive the firm to send messages that each generate some irredundant word-of-mouth effect.

To see concretely the role of open irredundance in relation to minimum dominance, consider again the four-consumer line network and the minimum dominating set marked in Figure 5. This minimum dominating set is not open irredundant, in the sense that one of

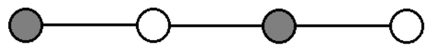


Figure 5: An open redundant minimum dominating set

its constituent nodes can be dropped without reducing the number of nodes who remain neighbors to at least one node in the set. Such a minimum dominating set cannot form a strategy for a firm in symmetric equilibrium when the cost of informing a consumer is high. Consider firm 2's best response if firm 1 was to inform this set. If firm 2 also informs this set, it will earn $\frac{1}{2}v$ for all four consumers, since all will learn of the two firms simultaneously (either today or tomorrow), at a cost of $2c$. If firm 2 instead informs only consumer 3, it earns $\frac{1}{2}v$ for only three consumers, since consumer 1 never learns of firm 2, at a cost of c. When $c > \frac{1}{2}v$, this minimum dominating set is not a best response to itself, and so cannot be played in a symmetric equilibrium. The underlying issue is that a open redundant minimum dominating set admits the possibility of reducing the number of costly direct messages sent without the indirect consequence of reducing word-of-mouth.

Once again, industry profits are lower than in the monopolistic case, and again this is due to an increased volume of advertising. However, in this case with low intensity of competition the symmetry of the competitors' strategies means that the increased volume is more starkly 'wasteful' in the sense of being a duplication of effort. Consider a setting that shares some features with this case: the milk industry in the United States. The pervasive existence of price supports (Manchester and Blayney (2001)) means that the intensity of competition among individual producers is somewhat constrained, despite the product being, to a first approximation, homogeneous. The existence of industry-wide advertising cooperatives ('Got Milk?') in this industry reflects the room for a potential 'collusive' marketing agreement in a such a setting, to overcome wasteful replication of marketing effort.

Finally we can again establish comparative statics on the cost of a message and the number of links:

Corollary 3. *In the duopolistic case with low intensity of competition,*

a. *for low c the marginal benefit to each firm of adding a link to g is zero, and for high c, in symmetric equilibria the marginal benefit to each firm of adding a link to g is either zero or c, depending on the location of the link; and*

b. *each firm's costs and profit can be increasing or decreasing in c.*

These results again invoke that the domination number is never increasing and decreases by at most one when a link is added. Even so, the number of links in the network affects profits only in the case when the cost of informing a consumer is high enough to preclude each firm from informing everyone in equilibrium. The effect on profits in that case is the same as for the monopolistic firm; if the extra link reduces the domination number, in symmetric equilibria each firm receives the same revenue as before but cost is lowered.

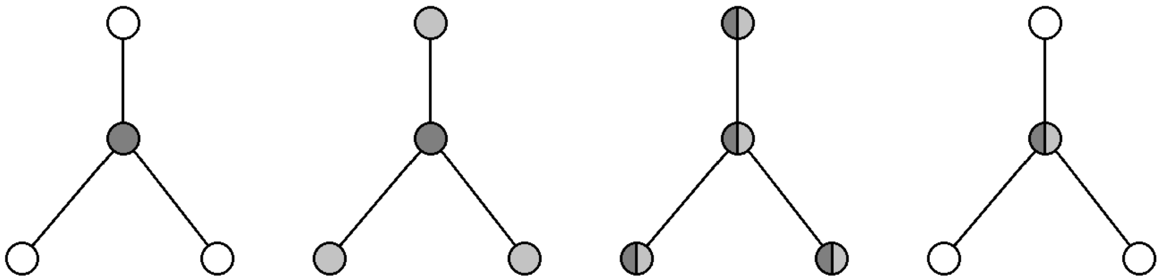
The non-monotonicity in part b. of Corollary 3 is a consequence of equilibrium selection as a function of the parameter c. Costs are increasing and profits decreasing in the intuitive way as c increases so long as c is in the range that either Proposition 3 or 4 applies. But near the threshold that selects one or the other case, an increase in c can decrease the cost of targeting and increase profits for each firm by switching from the equilibrium in which each firm informs all consumers to one in which each firm informs at most $IR(g)$ consumers. In Section 6 we return to this result in the context of intermediaries who may set c.

5 Optimal influence in families of networks

The previous sections describe the nature of equilibrium firm behavior in the game of disseminating information to some arbitrary network of consumers. The goal of this section is to relate these results to the targeting patterns they imply in some simple classes of network structure.

Figure 6 shows a 4-consumer star network[11]. The minimum dominating set of a star net-

[11] The two types of shaded node represent nodes targeted by one or the other firm, 'split' nodes are targeted by both firms, and blank nodes are not targeted



(a) Monopolist (b) Duopoly, high competition (c) Duopoly, low competition, low cost (d) Duopoly, low competition, high cost

Figure 6: Targeting in a star graph

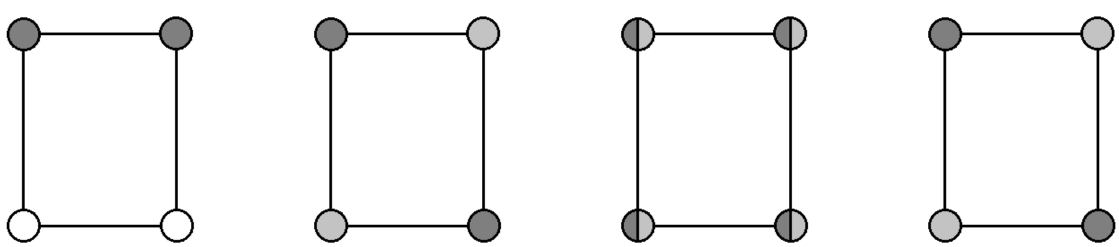
work is a single node - the hub - and so the most parsimonious way to inform all consumers is to directly inform the hub, with all others learning by word-of-mouth. This, then, is the strategy employed by the monopolist (Figure 6a), and, since this set is also open irredundant, a similar strategy can be accessed in equilibrium by a duopolist, provided that price competition is not 'too fierce' and the cost of sending messages is sufficiently high (Figure 6d).

In the remaining two cases each duopolist sends, on average, more than one message and the network is saturated with messages. In the case with gentle price competition and a low cost of sending messages, this saturation sees each node receive a direct message from both firms (Figure 6c), while in the case with Bertrand competition over common nodes, each node receives a message but from only one firm (Figure 6b). The segmentation of the network in this latter case is such that one firm informs the hub and the other the periphery. We therefore see equilibrium strategies that are able to make profitable use word-of-mouth only when competition is tempered both by low price competition and high costs; if firms compete fiercely, the structure of word-of-mouth affects targeting behavior but neither firm ultimately captures a consumer who learned of them by word-of-mouth. Further, as noted earlier, the pattern of segmentation induced by fierce price competition can yield very unequal outcomes for the two duopolists.

These patterns of equilibrium influence in Figure 6 carry to the family of star networks with more than two nodes, but also to the family of core-periphery networks in which each member of the core has at least one private neighbor in the periphery. Such a network has a set of nodes that are completely connected (the core) and a distinct set of nodes who each have one neighbor who is in the core (the periphery). This means that the analogs of 6a and 6d see the firm(s) target all nodes in the core, while the analog of 6b has one firm target the core and the other the periphery.

In Figure 6 we see an ordering of the total number of messages sent in each competitive regime. Of the duopolistic cases, the greatest number of total messages sent under complete saturation by both firms in the low competition, low cost case; the next greatest number of messages is in the segmented case under high competition; the smallest number of messages is in the low competition, high cost case. In all cases the duopoly together sends more messages than the monopolist.

This relationship between competitive regime and the number of messages sent to achieve optimal influence in equilibrium carries in all families of networks, with the caveat that for some networks the symmetric equilibrium in the low competition, high cost duopoly is not unique. Figure 7 considers the 4-consumer ring network; in this case equilibria in the low competition, high cost duopoly can be symmetric on the same set as the monopolist, or, as illustrated in 7d, asymmetric in the same manner as highly competitive segmentation. This is in general true of any network in which the complement of an open irredundant minimum dominating set is also an open irredundant minimum dominating set.



(a) Monopolist

(b) Duopoly, high competition

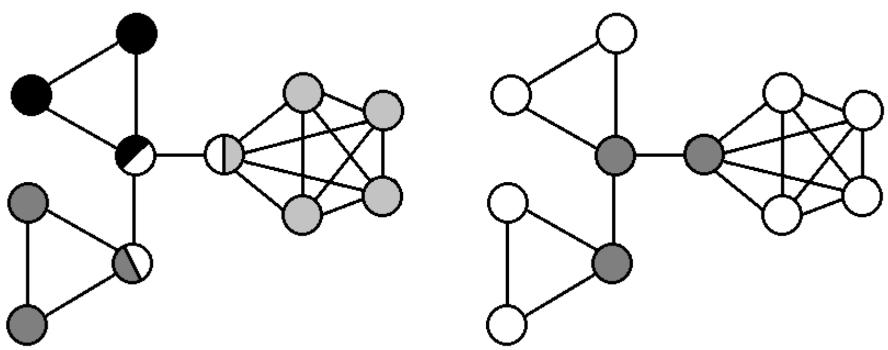
(c) Duopoly, low competition, low cost

(d) Duopoly, low competition, high cost

Figure 7: Targeting in a ring graph

In the cases in which targeting exploits word-of-mouth, networks with lower domination number are better for the firm since they allow more parsimonious targeting. In this example of the 4-ring network, firm profit is lower in the cases in which targeting can exploit word-of-mouth in equilibrium (7a and 7d) than it was in the 4-star network, since the domination number of the 4-ring network is higher, despite the fact that there are more links in the ring than the star. This further illustrates that while adding a link to a given network structure is never bad for the firm, it is not in general the case that the firm necessarily does better in a network with more links, and the precise architecture of the network is therefore relevant.

Finally, a recurring theme of the results above has been that in cases in which word-of-mouth can be exploited in equilibrium, optimal influence does not equate to targeting the best-connected consumers. A specific class of networks that are particularly sensitive to this distinction is networks with *cliques*. In Figure 8a there are three completely connected subgraphs: for example, the upper-left three nodes (with the darkest shading) are completely connected to each other. A completely connected component is called a clique.



(a) Three cliques (b) MDS marked

Figure 8: A network of three cliques

Figure 8b illustrates a minimum dominating set of this block graph: each clique receives one message. Since the cliques are completely connected, one message is enough to dominate the whole clique. Notice that several of the 'best-connected' consumers - in the largest clique, on the right of the figure - are untargeted. Networks with large (and variously sized) cliques will have a particularly pronounced gap between the success of a strategy that targets the consumers with the most connections and the optimal targeting identified here.

This is especially relevant since the empirically important family of 'small-world networks' (see for example Watts and Strogatz (1998), Strogatz (2001)) is characterized by a greater propensity to cliques than either random graphs or structurally ordered graphs. Small worlds are characterized (informally) by a large population, a small maximal degree, a short average path length between two randomly selected nodes, but high clustering - people have a small number of connections relative to the population, but two people who have a mutual friend are more likely to be connected to each other than they would be in a random graph. This type of network structure is anecdotally familiar and has been identified in a broad variety of settings (see Boccaletti, Latora, Moreno, Chavez, and Hwang (2006) for a thorough review). The distinction between optimal influence and location-blind targeting in settings that share features with the model presented above is thus likely to be pervasive in the types of networks most commonly observed in real-world settings.

6 Information intermediaries

Interpreting real settings in the context of the model requires us to understand the possibilities and constraints on the ability of the informer to reach subsets of consumers. As discussed in the beginning of the paper, the model literally describes a situation in which the informer can precisely construct or observe a relevant communication network operating at some level over its market and send direct marketing messages to a subset of those consumers. In many applications, disseminating information is fuzzier, and best achieved through information intermediaries - media, advertising, or public relations firms, for example. This section therefore briefly considers how the ideas of the model can translate to settings with looser assumptions on the availability of precise targeting by considering the problem for information intermediaries.

An intermediary selling direct marketing services can in principle negotiate with a client a contract that defines a price to inform a given subset of the market. The extent to which this reachable audience is also a strategic variable may itself vary. For some intermediaries, the subset of the market that they can offer to a client is fixed; a newspaper's audience, for example, is fixed in the short run. For others, perhaps an online advertising platform like Google's AdWords, it may be possible to offer a menu of various subsets of the target population.

The market structure in the informer's industry and the shape of the communication network among the target population can plausibly associate to various approaches by an information intermediary. On price, Propositions 3 and 4 admit the possibility that the volume of direct marketing desired by an informer can be *negatively* associated with its price: as the price of direct marketing falls, there is an incentive for competing firms to switch from a strategy that targets the word-of-mouth-maximizing network locations to one that saturates the network with messages. A profit-maximizing intermediary with discretion over the set of consumers it can reach may thus prefer to set a low price for its services in order to induce the 'arms race' that drives clients in weakly competitive industries to mass-marketing. Similarly, all else equal, this effect could lead an intermediary tied to a mass audience to rationally set a lower price per impression for advertising services than an intermediary tied to a niche audience. The direct competitors to mass media outlets for an advertising contract with a firm that has significant market power are therefore not only other mass media outlets, but include higher price-per-impression niche media outlets that might superficially seem better tailored to niche products.

On the audience served by the intermediary, for an industry characterized by a high intensity of competition, Proposition 2's segmentation result drives firms to market to distinct but intermingled sets of consumers. This could correspond to competitors advertising in different newspapers or websites, for example; competing newspapers serve approximately distinct readerships, and while communication channels may be richer among than between these readerships, the between-channels are not trivial. This suggests that natural partners for an information intermediary that serves an audience that has little overlap with that served by other intermediaries are firms in highly competitive industries; the fiercely competitive model predicts that such a firm will seek to deliver its messages to an audience distinct from its competitor.

Propositions 3 and 4, by contrast, suggest that low intensity of competition can beget symmetric direct marketing strategies. The burden on an information intermediary or media market to offer a variety of outlets that capture distinct audiences is correspondingly less. Instead, mass media can more valuable to the informing firm. In this fashion the problem for an information intermediary seeking to fill advertising space is informed by the audience it reaches: the characteristics of firms that will find a given media outlet most valuable depend on the nature of that firm's industry.

7 Concluding comments

This paper develops and analyzes a theoretical model of the problem for a firm seeking to disseminate information when it is able to locate direct marketing at locations in a communication network among its population to exploit word-of-mouth. A pervasive prediction of the analysis, across the various competitive structures considered above, is of dispersed rather than concentrated targeting of direct marketing. In weakly competitive settings, the role of word-of-mouth in amplifying direct messages precludes sending many messages to any one network locality. In more competitive settings this incentive interacts with the incentive for fierce competitors to segment the market rather than overlap to predict dispersed but *intermingled* targeting of direct messages. Translating this analysis from direct marketing to advertising via intermediaries implies that for a firm trying to disseminate information to some population, media with broad and shallow reach across that population can be more valuable advertising outlets than media with deep capture of a particular sub-population.

The boundaries of the model, imposed primarily by tractability, suggest areas in which it could be extended. First, the model considers purely informative marketing, and abstracts from persuasive advertising and persuasive word-of-mouth. Second, the model is non-specific on how the information is used by those who receive it, assuming simply that a firm does better the more consumers hear of it before hearing of its competitors. Third, the exact nature of information transmission is simplified for tractability to be (i.) strictly local, (ii.) non-optional, and (iii.) non-strategic. Fourth, although Section 6 begins to explore the relationship between the literal precise targeting of the model and more general targeting mechanisms, the model assumes very fine targeting. While the model is suggestive of the relationship between the competitive structure of an industry, the structure of consumers' communication network, and the menu of available marketing channels, these limitations imply that to translate these insights to a given setting requires work to understand the exact nature of information processing and transmission in that setting.

References

Banerji, A., and B. Dutta (2009): "Local Network Externalities and Market Segmentation," *International Journal of Industrial Organization*, 27(5), 605–614.

Boccaletti, S., V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang (2006): "Complex networks: Structure and dynamics," *Physics Reports*, 424(4-5), 175–308.

Bollobás, B., and E. Cockayne (1979): "Graph-theoretic parameters concerning domination, independence, and irredundance," *Journal of Graph Theory*, 3(3), 241–249.

——— (1984): "The irredundance number and maximum degree of a graph," *Discrete Mathematics*, 49(2), 197–199.

Butters, G. R. (1977): "Equilibrium Distributions of Sales and Advertising Prices," *The Review of Economic Studies*, 44(3), 465–491.

Cockayne, E., S. Hedetniemi, and P. Slater (1978): "Properties of Hereditary Hypergraphs and Middle Graphs," *Canadian Mathematical Bulletin*, 21, 461–468.

Coleman, J. S., E. Katz, and H. Menzel (1966): *Medical Innovation*. Bobbs-Merrill.

Domingos, P., and M. Richardson (2001): "Mining the Network Value of Customers," *Proc. 7th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pp. 57–66.

Farley, A., and N. Schacham (1983): "Senders in broadcast networks: open irredundancy in graphs," *Congressus Numerantium*, 38, 47–57.

Fellows, M., G. Fricke, S. Hedetniemi, and D. Jacobs (1994): "The Private Neighbor Cube," *SIAM Journal on Discrete Mathematics*, 7(1), 41–47.

Galeotti, A., and S. Goyal (2009): "Influencing the influencers: a theory of strategic diffusion," *RAND Journal of Economics*, 40(3), 509–532.

Galeotti, A., and J. Moraga-Gonzalez (2008): "Segmentation, advertising and prices," *International Journal of Industrial Organization*, 26(5), 1106–1119.

Garey, M. R., and D. S. Johnson (1979): *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman.

Godes, D., and D. Mayzlin (2004): “Using Online Conversations to Study Word-of-Mouth Communication,” *Marketing Science*, 23(4), 545–560.

——— (2009): “Firm-created Word-of-mouth Communication: Evidence from a Field Test,” *Marketing Science*, 28(4), 721–739.

Haynes, T. W., and M. A. Henning (2003): “Changing and unchanging domination: a classification,” *Discrete Mathematics*, 272(1), 6579.

Iyengar, R., C. Van den Bulte, and T. W. Valente (2011): “Opinion Leadership and Social Contagion in New Product Diffusion,” *Marketing Science*, 30(2).

Iyer, G., D. Soberman, and J. M. Villas-Boas (2005): “The Targeting of Advertising,” *Marketing Science*, 24(3), 461–476.

Jackson, M. O. (2008): *Social and Economic Networks*. Princeton University Press.

Katz, E., and P. F. Lazarsfeld (1955): *Personal Influence: The Part Played by People in the Flow of Mass Communications*. Glencoe, IL: Free Press.

Kempe, D., J. Kleinberg, and Éva Tardos (2003): “Maximizing the Spread of Influence through a Social Network,” *Proc. 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pp. 137–146.

Manchester, A. C., and D. P. Blayney (2001): “Milk Pricing in the United States,” *USDA Agriculture Information Bulletin*, (No. 761).

Ore, O. (1962): *Theory of Graphs*. Providence, American Mathematical Society, Colloquium publications v.38.

Richardson, M., and P. Domingos (2002): “Mining Knowledge-Sharing Sites for Viral Marketing,” *Proc. 8th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pp. 61–70.

Rogers, E. M., and D. G. Cartano (1962): "Methods of Measuring Opinion Leadership," *Public Opinion Quarterly*, 26, 435–441.

Roy, S. (2000): "Strategic segmentation of a market," *International Journal of Industrial Organization*, 18(8), 1279–1290.

Strogatz, S. H. (2001): "Exploring complex networks," *Nature*, 410, 268–276.

Tucker, C. (2008): "Identifying Formal and Informal Influence in Technology Adoption with Network Externalities," *Management Science*, 54(12), 2024–2038.

Van den Bulte, C., and G. L. Lilien (2001): "*Medical Innovation* Revisited: Social Contagion vesus Marketing Effort," *The American Journal of Sociology*, 106(5), 1409–1435.

Watts, D., and P. S. Dodds (2007): "Influentials, Networks, and Public Opinion Formation," *Journal of Consumer Research*, 34(4), 441–458.

Watts, D. J., and S. H. Strogatz (1998): "Collective dynamics of 'small-world' networks," *Nature*, 393, 440–442.

Zubcsek, P. P., and M. Sarvary (2011): "Advertising to a Social Network," *Quantitative Marketing and Economics*, 9(1), 71–107.

A Proofs

Proposition 1

Proof. Assume not, so that $S_m \neq S_m^*$. If $s_m < s_m^*$, some consumers will never learn about the firm, since they are more than one link away from a consumer who is directly informed. Informing those consumers directly, by selecting some $\hat{S} \supset S_m$, is profitable since $p > c$. If instead $s_m > s_m^*$, then all consumers will indeed learn about the firm and buy, but this could be achieved at lower cost by reducing s_m by selecting some minimum dominating set of g. Thus $S_m \neq S_m^*$ was not profit-maximizing. □

Proposition 2

Proof. Take three steps:

a) First we will prove that S_k and S_l must be disjoint in equilibrium. Suppose not; then, there must be at least one consumer a common to both sets. A deviation by firm k to the set $S_k - a$ saves c and costs 0. S_k is not a best response to S_l, so (S_k, S_l) was not an equilibrium.

b) Second, we will prove that $S_k \cup S_l = A$ in equilibrium. Suppose not, so that $\exists b$ such that $b \notin S_k, S_l$. W.l.o.g, it must be the case that either $b \notin \Omega(S_k), \Omega(S_l)$, $b \in \Omega(S_k), \Omega(S_l)$ or $b \notin \Omega(S_k)$, $b \in \Omega(S_l)$; in any of these cases, a deviation by firm k to $S_k + b$ yields v and costs c, which is profitable by assumption. S_k was therefore not a best response to S_l, so the pair $S_k \cup S_l \neq A$ is not an equilibrium.

c) Finally, we will prove that S_k and S_l must be dominating sets in equilibrium. Assume not, so that S_k is not a dominating set. This means that there exists a consumer $d \in S_l$ such that $d \notin S_k, \Omega(S_k)$. Since the graph is connected and since by b) $S_k \cup S_l = A$ it must be that $d \in S_l, \Omega(S_l)$ and $\nexists e \in \Omega(d), e \notin S_k, S_l$. Therefore a deviation by firm l to $S_l - d$ saves c, but loses nothing since consumer d will be served by firm l at price p in round 2. S_l was not a best response to S_k, so the pair cannot constitute an equilibrium.

□

Proposition 3

First we will show that such an equilibrium exists:

Lemma 1. *$S_1 = A$, $S_2 = A$ is a Nash equilibrium for any network g when $c < \frac{1}{2}v$.*

Proof. Fix $S_j = A$. If firm k selects $S_k = A$, all consumers will be directly informed by both firms in stage 1, so each firm receives value $\frac{1}{2}v$ for each individual. The payoff to firm k is:

$$\pi_k = n(\frac{1}{2}v - c) \tag{A.1}$$

If firm k instead selects $S_k = M \subset A$, those $n - m$ consumers will learn of firm 1 but not firm 2 in stage 1, and yield value zero to firm k. Firm k's payoff is:

$$\pi_k = m(\frac{1}{2}v - c) \tag{A.2}$$

If $c < \frac{1}{2}v$, this is always less than the payoff to $S_k = A$. $S_k = A$ is thus a best response to $S_j = A$, and so the pair $S_1 = A$, $S_2 = A$ constitutes a Nash equilibrium. □

Second, we establish uniqueness:

Lemma 2. *When $c < \frac{1}{2}v$, $S_1 = A$, $S_2 = A$ is the unique Nash equilibrium for any network g.*

Proof. If at least one firm does not send messages to A, there are three possibilities:

i. $S_j = S_k\emptyset$: $\pi_j = 0$; if firm j instead played $S_j = A$, $\pi_j = nv - nc$, which is greater than zero by primitive assumption.

ii. $|S_j| < |S_k|$, $S_k \subseteq A$: firm j earns zero from $i \in S_k, \notin S_j$ since that consumer learns of k before j. If firm j instead played $S_j + i$, it pays c more and earns $\frac{1}{2}v$ from i since then i learns of the two firms at the same time. Since $\frac{1}{2}v > c$, this yields a higher payoff than S_j.

iii. $S_j = S_k \subset A$: firm j earns $\frac{1}{2}v$ from $l \in \Omega(S_j)$ since that consumer leans of both firms at the same time tomorrow. If firm j instead played $S_j + l$, it pays c more and earns

$\frac{1}{2}v$ more from l since l will then learn of j before k. Since $\frac{1}{2}v > c$, this yields a higher payoff than S_j.

In each case the proposed pair is not an equilibrium; there is no Nash equilibrium that does not have $S_1 = A$, $S_2 = A$. □

This completes Proposition 3.

Proposition 4

Part a.

In equilibrium neither firm chooses a redundant set, and so each firm sends at most $IR(g)$ messages.

Proof. Assume not, so that there exists an equilibrium in which firm j chooses to inform a redundant set R of consumers. Since R is a redundant set, there exists at least one i such that $\{R + \Omega(R)\} = \{R - i + \Omega(R - i)\}$. Consider a deviation by firm j to the set $R - i$.

For all vertices in R, firm j captures the same value as before, since those consumers continue to learn about firm j today when they receive a message. For all vertices in $\Omega(R)$, firm j again captures the same value as before, since those consumers continue to learn about firm j tomorrow by word-of-mouth. For all vertices not in R or $\Omega(R)$, firm j continues to capture zero.

For i, consider separately three exhaustive cases.

i. $i \in S_k$. When $S_j = R$, i learns of both firms today when receiving a message from each, and j receives $\frac{1}{2}v$ from i. When $S_j = R - i$, i learns of firm k before j and so j receives zero from i.

ii. $i \notin S_k, \in \Omega(S_k)$. When $S_j = R$, i learns of j today and k tomorrow, and j receives v from i. When $S_j = R - i$, i learns of j and k tomorrow by word-of-mouth and so j receives $\frac{1}{2}v$ from i.

iii. $i \notin S_k, \notin \Omega(S_k)$. When $S_j = R$, i learns of j today and never learns of k, and j receives v from i. When $S_j = R - i$, i learns of j tomorrow and never learns of k and so j receives v from i.

In each case firm j loses at most $\frac{1}{2}v$ from i. Since informing $R - i$ saves c, the value obtained from all other nodes is identical, and $c > \frac{1}{2}v$, firm j earns a greater payoff from $R - i$ than from R regardless of the choice of firm k. Thus R was not a best response by j and so cannot be part of an equilibrium. □

Part b.

In equilibrium each consumer learns of at least one firm by tomorrow.

Proof. Assume not, so that there exists an equilibrium at S_j, S_k such that there is some consumer $i \notin S_j, S_k, \Omega(S_j), \Omega(S_k)$. A deviation by firm j to $S_j + i$ costs an extra c and yields an extra v, since i will learn of firm j today and not learn of firm k. Since $v > c$ by assumption, the payoff to firm j from $S_j + i$ is greater than that from S_j. S_j was therefore not a best response to S_k and so the original pair was not an equilibrium. □

Part c.

There is a symmetric equilibrium at $S_j = S_k = S$ if and only if S is an open irredundant minimum dominating set.

Proof. First we show that $S \in BR(S)$ when $S \in OIMDS(g)$ (since $OIMDS(g)$ is nonempty such an S exists always).

Fix $S_k = S \in OIMDS(g)$. If $S_j = S$, each consumer learns of both firms at the same time and so j's payoff is $\frac{1}{2}nv - |S|c$. Consider instead some $S_j = T \neq S$. There are five possibilities for each consumer in A:

i. Each $i \in T, S$ continues to learn of both firms today and so the payoff to j from each such consumer is unchanged.

ii. Each $i \notin T, S, \in \Omega(T)$ continues to learn of both firms tomorrow and so the payoff to j from each such consumer is unchanged.

iii. Each $i \notin T, S, \Omega(T)$ now does not learn of j and so j loses $\frac{1}{2}v$ for each such consumer.

iv. Each $i \in S, \notin T$ now learns of firm j after learning of firm k and so j loses $\frac{1}{2}v$ but saves c for each such consumer.

v. Each $i \in T, \notin S$ now learns of firm j before firm k and so j gains $\frac{1}{2}v$ but pays c for each such consumer.

If $|T| > |S|$ then the number of consumers in case v. is greater than the number in case iv.; since $\frac{1}{2}v < c$, the payoff to j in this case is strictly lower than if $S_j = S$. If $|T| = |S|$ then the gain and loss in cases iv. and v. are exactly offsetting. Then if there are any consumers in case iii. the payoff to j is strictly lower than if $S_j = S$, or if there are no consumers in case iii. (that is, if $T \in OIMDS(g)$) the payoff to j is equal to that if $S_j = S$.

Finally, if $|T| < |S|$, the number of consumers in case iv. is greater than the number in case v., but since $S \in OIMDS(g)$, each consumer in S has at least one private neighbor in S^C. Consider the private neighbors to the consumers in $S - T$, $pn(S - T)$. Each consumer in $pn(S - T)$ is either in the set T and so is in case v., in which case the loss from informing that private neighbor offsets the gain from not having informed one consumer in $S - T$, or they are not in T and so are in case iii., in which case j loses a further $\frac{1}{2}v$. But since $|T| < |S|$ it must be that fewer of these private neighbors are in T than are not, and so since $c < v$, the payoff to j is strictly lower than if $S_j = S$.

Thus $S \in BR(S)$ and so there exists a symmetric equilibrium in which $S_j = S_k = S \in OIMDS(g)$.

Next we show that there are other no symmetric equilibria.

i. Assume an equilibrium exists at $S_j = S_k = Z \notin DS(g)$, where $DS(g)$ is the set of dominating sets of g. Since Z is not a dominating set, there exists some consumer $i \notin Z, \Omega(Z)$ who never learns of either firm. A deviation by firm j to $Z + i$ costs an extra c and yields value v, since i will learn of firm j today and not learn of firm k. $S_j = Z$ is therefore not a best response to $S_k = Z$ and the original pair is not an equilibrium; there is no symmetric equilibrium at a set that is not dominating.

ii. Assume an equilibrium exists at $S_j = S_k = Y \in DS(g), \notin OIDS(g)$. In such an equilibrium, since $Y \in DS(g)$, all consumers in Y learn of both firms today and all remaining consumers learn of both firms tomorrow, and so $\pi_j(Y) = \frac{1}{2}nv - yc$. Since Y is not an open irredundant dominating set, there exists some consumer $l \in Y$ such that $\Omega(Y - l) = \Omega(Y)$. A deviation by firm j to $Y - l$ means that $y - 1$ consumers learn of both firms j today, 1 consumer learns only of firm k today, and $n - y$ consumers learn of both firms tomorrow. This yields $\pi_j(Y - l) = \frac{1}{2}(n-1)v - (y-1)c$, which is greater than $\pi_j(Y)$ since $\frac{1}{2}v < c$. $S_j = Y$ is therefore not a best response to $S_k = Y$ and the original pair is not an equilibrium; there is no symmetric equilibrium at a set that is not open irredundant and dominating.

iii. Assume an equilibrium exists at $S_j = S_k = X \in OIDS(g), \notin OIMDS(g)$. In such an equilibrium, each consumer learns of both firms at the same time and so each firm earns $\frac{1}{2}nv - |X|c$. Consider a deviation by firm j to $W \in OIMDS(g)$. Each consumer in both X and W or in neither X nor W in both cases learns of both firms at the same time, and so firm j continues to earn $\frac{1}{2}v$ for each such consumer. Each consumer in X but not W now learns of firm k today and j tomorrow, and so firm j loses $\frac{1}{2}v$ but saves c on each such consumer. Finally, each consumer in W but not in X now learns of firm j today and k tomorrow, and so firm j gains $\frac{1}{2}v$ but spends c on each such consumer. Since $W \in OIMDS(g)$ and $X \in OIDS(g), \notin OIMDS(g)$, $|W| < |X|$ and so the number of consumers in X but not W is larger than the number of consumers in W but not X; since $\frac{1}{2}v < c$, when $S_j = W$ yields a higher payoff for j than $S_j = X$. $S_j = X$ is therefore not a best response to $S_k = X$ and the original pair is not an equilibrium; there is no symmetric equilibrium at a set that is not open irredundant minimum dominating.

□